hep-th 9511220
OU-HET 222
November, 1995

Macroscopic n-Loop Amplitude for Minimal Models Coupled to Two-Dimensional Gravity — Fusion Rules and Interactions — [1]

M. Anazawa † [2]
and
H. Itoyama †

† Department of Physics,
Graduate School of Science, Osaka University,
Toyonaka, Osaka, 560 Japan

Abstract

We investigate the structure of the macroscopic n-loop amplitude obtained from the two-matrix model at the unitary minimal critical point $(m+1, m)$. We derive a general formula for the n-resolvent correlator at the continuum planar limit whose inverse Laplace transform provides the amplitude in terms of the boundary lengths ℓ_i and the renormalized cosmological constant t. The amplitude is found to contain a term consisting of $\left(\frac{\partial}{\partial t}\right)^{n-3}$ multiplied by the product of modified Bessel functions summed over their degrees which conform to the fusion rules and the crossing symmetry. This is found to be supplemented by an increasing number of other terms with n which represent residual interactions of loops. We reveal the nature of these interactions by explicitly determining them as the convolution of modified Bessel functions and their derivatives for the case $n = 4$ and the case $n = 5$. We derive a set of recursion relations which relate the terms in the n-resolvents to those in the $(n-1)$-resolvents.

[1] This work is supported in part by Grant-in-Aid for Scientific Research (07640403) and by the Grand-in-Aid for Scientific Research Fund (2690) from the Ministry of Education, Science and Culture, Japan.

[2] JSPS fellow

I. Introduction and Conclusion

Matrix models provide an arena in which the notion of integrability is realized as noncritical string theory. At the same time, they produce efficient computation of some quantities which would be very formidable in the continuum framework. Computation of macroscopic loop amplitudes[1, 2, 3] [4] demonstrates this fact most explicitly: the boundary condition which is hard to solve in the continuum framework [5] turns out to be related to the most natural quantity in matrix models. Let us begin with recalling this.

A crude correspondence of matrix models with path integrals of noncritical strings tells us that the connected part of the correlator given by averaging over matrix integrals of the product of singlet correlators

$$<< tr\hat{M}^{q_1} tr\hat{M}^{q_2} \cdots tr\hat{M}^{q_n} >>_{N,conn} \tag{1.1}$$

is an n-punctured surface swept by a noncritical string. To turn these punctures into holes of a macroscpic size, one first introduces a fixed loop length at the i-th boundary by $\ell_i = aq_i$. We are naturally led to consider the limiting procedure

$$\mathcal{A}_n\left(\ell_1, \ell_2, \cdots \ell_n\right) \equiv \lim_{q_i\to\infty, a\to 0,} \lim_{N\to\infty} \frac{1}{\kappa^{n-2}} << tr\hat{M}^{q_1} tr\hat{M}^{q_2} \cdots tr\hat{M}^{q_n} >>_{N,conn} \tag{1.2}$$

which defines the macroscopic n-loop amplitude.[3] Here κ is the renormalized string coupling and a is an auxiliary parameter which plays the role of a cutoff.

An equivalent and more efficient procedure is to consider the correlator consisting of the product of n-resolvents $<< \prod_{i=1}^{n} tr\frac{1}{p_i - \hat{M}} >>_{N,conn}$, to pick its most singular piece and finally to carry out the inverse Laplace transforms over p_i's. This in turn means

$$\mathcal{A}_n\left(\ell_1, \ell_2, \cdots \ell_n\right) = \left(\prod_{j=1}^{n} \mathcal{L}_j^{-1}\right) \lim_{a\to 0} \lim_{N\to\infty} \frac{a^n}{\kappa^{n-2}} << \prod_{i=1}^{n} tr\frac{1}{p_i - \hat{M}} >>_{N,conn} \quad . \tag{1.3}$$

Here, $\mathcal{L}_j^{-1}$ denotes the inverse Laplace transform with respect to ζ_i such that $a\zeta_i = p_i - p_i^*$ and p_i^* denotes the critical value of p_i. In this paper, we will carry out this procedure in depth at the $(m+1, m)$ critical point realized by the symmetric potential of the two-matrix model. [4] Here $\kappa \equiv \frac{1}{Na^{2+1/m}}$.

[3] See later sections for more of the definitions.

[4] For some of the recent works on the two-matrix models, see, for instance, [12].

In the next section, we evaluate the connected part of the correlator consisting of the product of n-resolvents for large N just mentioned above and derive a general formula for this object in the continuum planar limit. We exploit the planar solution to the Heisenberg algebra and its parametrization provided by [6]. Our formula contains a term distinguishable from others, namely the one which is expressible as the total $\left(\frac{\partial}{\partial t}\right)^{n-3}$ derivatives. Here t denotes the renormalized cosmological constant. This structure is familiar from the case of pure two-dimensional gravity. This term is, however, found to be supplemented, for $n \geq 4$, by an increasing number of other terms with n. This latter structure testifies to the existence of interactions which cannot be captured by the naive notion of operator product expansion for microscopic loop operators: the macroscopic loop operator will be expanded by these. For that reason, these interactions may be referred to as contact interactions.

In section III, we consider the $\left(\frac{\partial}{\partial t}\right)^{n-3} \cdots$ term. We are successful in representing this terms as the summations over $2n-3$ indices with its summand in a form of n factorized products. These summations are found to conform to the fusion rules and the crossing symmetry for the dressed primaries of the unitary minimal conformal field theory [7]. Using the formula for the inverse Laplace transform found in [8],

$$\mathcal{L}^{-1}[\frac{\partial}{\partial \zeta}\frac{\sinh k\theta}{\sinh m\theta}] = -\frac{M\ell}{\pi}\sin\frac{k\pi}{m}\ K_{k/m}(M\ell) \equiv -\frac{M\ell}{\pi}\underline{K}_{k/m}(M\ell) \ . \tag{1.4}$$

we determine the complete form for this part of the amplitude in terms of the boundary lengths ℓ_i $i = 1, \cdots, n$. The answer reads as

$$\begin{aligned}&\mathcal{A}_n^{fusion}(\ell_1, \ell_2, \cdots \ell_n)\\ &\quad = -\frac{1}{m}\left(\frac{1}{m+1}\right)^{n-2}\left(\frac{\partial}{\partial t}\right)^{n-3}\left[t^{-1-\frac{(n-2)}{2m}}\sum_{\mathcal{D}_n}\prod_{j=1}^{n}\frac{M\ell_j}{\pi}\underline{K}_{1-k_j/m}\left(M\ell_j\right)\right]\end{aligned} \tag{1.5}$$

The case $n = 3$ has been briefly reported in [9]. In section IV, we consider the remaining pieces in the formula which represent the residual interactions of loops. For the case $n = 4$ and the case $n = 5$, we have succeeded in expressing these in terms of the convolution of modified Bessel functions and their derivatives. We, therefore, obtain the complete answer for $\mathcal{A}_4\left(\ell_1, \cdots, \ell_4\right)$ and the one for $\mathcal{A}_5\left(\ell_1, \cdots, \ell_5\right)$, which are eq. (4.14) and eq. (4.22) respectively. Although it is not unlikely that one can determine the full amplitude this way for arbitrary n, the proof remains elusive. We will finish with a few remarks concerning with the properties of these residual interactions.

In Appendix A, we derive a set of recursion relations which are used to evaluate the formula in section III. These recursion relations relate the expression of the terms appearing in the n-resolvent to those in the $(n-1)$-resolvent. These define, therefore, the n-loop amplitude in terms of $(n-1)$-loop amplitude through the inverse Laplace transforms albeit being implicit.

II. The n-Resolvent Correlator in Continuum Planar Limit

Consider in the two-matrix model the connected part of the correlator consisting of the product of n-resolvents at finite N:

$$<< tr\frac{1}{p_1-\hat{M}} tr\frac{1}{p_2-\hat{M}} \cdots tr\frac{1}{p_n-\hat{M}} >>_{N,conn} \quad . \tag{2.1}$$

Here, $(\hat{M}, \hat{\tilde{M}})$ are the matrix variables and p_i's are eigenvalue coordinates which, in the continuum limit, become Laplace-conjugate to loop lengths. We denote by $<< \cdots >>_{N,conn}$ the averaging with respect to the matrix integrations. It should be noted that this expression is at most $\left(\frac{1}{N}\right)^{n-2}$ due to the large N factorization of the correlator consisting of the product of singlet operators. In the second quantized notation [5], eq. (2.1) is expressible as

$$\begin{aligned} & {}_N< 0 \mid \prod_{i=1}^{n} : \int d\lambda_i b^\dagger(\lambda_i) \frac{1}{p_i-\lambda_i} b(\lambda_i) :\mid 0>_N \\ & = {}_N< 0 \mid \prod_{i=1}^{n} : B^\dagger_{k_i} B_{j_i} :\mid 0>_N \prod_{i=1}^{n} < k_i \mid \frac{1}{p_i-\hat{M}} \mid j_i> \quad , \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} b(\lambda) &= \sum_{j=0}^{\infty} <\lambda \mid j> B_j \quad , \quad b^\dagger(\lambda) = \sum_{j=0}^{\infty} < k \mid \lambda > B^\dagger_k \\ B_j \mid \Omega> &= 0 \quad , \quad j = 0, 1, 2, \cdots \\ \text{and} \qquad \mid 0>_N &\equiv \prod_{j=0}^{N-1} B^\dagger_j \mid \Omega> \quad . \end{aligned} \tag{2.3}$$

The normal ordering : ... : is with respect to the filled sea $\mid 0>_N$. We introduce a notation

$$\left[\frac{1}{p-\hat{M}}\right](z_i; \Lambda_i, \Lambda, N) \equiv \sum_{\delta} z_i^{\delta} < j_i - \delta \mid \frac{1}{p-\hat{M}} \mid j_i> \tag{2.4}$$

[5] See, for example, [10, 11]

$$\Lambda_i = j_i \Lambda / N = \Lambda + \Lambda \tilde{j}_i / N \quad . \tag{2.5}$$

The evaluation of ${}_N < 0 \mid \prod_{i=1}^{n} : B_{k_i}^{\dagger} B_{j_i} : \mid 0 >_N$ by the Wick theorem provides $(n-1)!$ terms of the following structure: each term is given by the product of n-Kronecker delta's multiplied both by a sign factor and by the product of n-step functions to ensure that the summations over the n-indices $\tilde{j}_1, \tilde{j}_2 \cdots$ and $\tilde{j}_n$ are bounded either from below (≥ 0) or from above(≤ -1). We denote this product by $\Theta(\tilde{j}_1, \tilde{j}_2, \cdots \tilde{j}_n; \sigma)$. These $(n-1)!$ terms are in one-to-one correspondence with the circular permutations of n integers $1, \cdots, n$, which we denote by $\mathcal{S}_n$. The σ is an element of $\mathcal{S}_n$. For large N, we find

$$\left(\frac{N}{\Lambda}\right)^{n-2} << \prod_{i=1}^{n} tr \frac{1}{p_i - \hat{M}} >>_{N,conn} \tag{2.6}$$

$$= \sum_{\tilde{j}_1, \tilde{j}_2, \cdots \tilde{j}_n} \sum_{\sigma \in \mathcal{S}_n} \Theta(\tilde{j}_1, \tilde{j}_2, \cdots \tilde{j}_n; \sigma) sgn(\sigma) \left(\prod_{j=1}^{n} \oint \frac{dz_j}{2\pi i} \right) \prod_{k=1}^{n} \frac{1}{z_k} \left(\frac{z_{\sigma(k)}}{z_k} \right)^{\tilde{j}_k}$$

$$\times \frac{1}{(n-2)!} \left(\sum_{i=1}^{n} \tilde{j}_i \frac{\partial}{\partial \Lambda_i} \right)^{n-2} \prod_{i'=1}^{n} \frac{1}{p_{i'} - \left[\hat{M}\right](z_{i'}; \Lambda_{i'})} |_{\Lambda_{i'} = \Lambda} + \mathcal{O}\left(1/N\right) \quad . \tag{2.7}$$

Note that in the large N limit, we can use $\frac{1}{p_i - \left[\hat{M}\right](z_i; \Lambda_i)}$ in place of $\left[\frac{1}{p_i - \hat{M}}\right](z_i; \Lambda_i, \Lambda, N)$ according to the same reason as stated in [9]. Here the 'classical' function is defined by

$$\left[\hat{M}\right](z_i; \Lambda_i) \equiv \lim_{N \to \infty} \sum_{\delta} z_i^{\delta} \langle j_i - \delta | \, \hat{M} \, | j_i \rangle \qquad , \quad \Lambda_i = j_i \Lambda / N \quad . \tag{2.8}$$

The $sgn(\sigma)$ denotes the signature associated with the permutation σ.

Let us define

$$m! D_m(z, z') \equiv \frac{1}{z} \sum_{\tilde{j} \geq 0} \tilde{j}^m \left(z'/z\right)^{\tilde{j}} = -\frac{1}{z} \sum_{\tilde{j} \leq -1} \tilde{j}^m \left(z'/z\right)^{\tilde{j}} \quad , \quad m = 0, \cdots \quad . \tag{2.9}$$

In the continuum limit we will be focusing from now on, it is sufficient to use

$$D_m(z, z') \approx \frac{1}{(z - z')^{m+1}} \equiv D_m(z - z') \quad . \tag{2.10}$$

Let $sgn_i(\sigma)$ be $+1$ or -1, depending upon whether the restriction on the summation over $\tilde{j}_i$ is bounded from below or from above respectively. It is not difficult to show

$$sgn(\sigma) \prod_{i=1}^{n} sgn_i(\sigma) = -1 \quad , \tag{2.11}$$

for any σ and n. The summations over $\tilde{j}_1, \tilde{j}_2 \cdots$ and $\tilde{j}_n$ can then be performed for all σ at once, leaving with this minus sign.

Now we turn to the integrations over z_i ($i = 1 \sim n$). The convergence on the geometric series leads to the successively ordered integrations of $z_i's$ for each σ. By simply picking up a pole of z_i at $\frac{1}{p_i - M(z_i;\Lambda_i)}$ for $i = 1 \sim n$ and using

$$\begin{aligned}
&\oint \frac{dz_i}{2\pi i} f(\cdots z_i, \cdots) \left(\frac{\partial}{\partial \Lambda_i}\right)^{\ell} \left(\frac{1}{p_i - \left[\hat{M}\right](z_i;\Lambda_i)}\right) \\
= \; & -\frac{\partial}{\partial(a\zeta_i)} \left(\frac{\partial}{\partial \Lambda_i}\right)^{\ell} \int^{z_i^*} dz_i f(\cdots z_i, \cdots) \quad , \quad \ell = 0, 1, \cdots \quad , \qquad (2.12)
\end{aligned}$$

we find that eq. (2.6) is written as

$$\left(\frac{N}{\Lambda}\right)^{n-2} << \prod_{i=1}^{n} tr \frac{1}{p_i - \hat{M}} >>_{N,conn} = \prod_{i=1}^{n} \left(-\frac{\partial}{\partial(a\zeta_i)}\right) \frac{-1}{(n-2)!} \tilde{\Delta}_n|_{\Lambda_i = \Lambda} \quad , \qquad (2.13)$$

where

$$\begin{aligned}
\frac{1}{(n-2)!}\tilde{\Delta}_n \; \equiv \; & \sum_{i_1}^{n} \left(\frac{\partial}{\partial \Lambda_{i_1}}\right)^{n-2} \int \cdots \int \sum_{\sigma \in \mathcal{S}_n} D_{n-2}([i_1 - \sigma(i_1)]) \prod_{j(\neq i_1)} D_0([j - \sigma(j)]) \\
+ \; & \sum_{i_1, i_2}^{n} \left(\frac{\partial}{\partial \Lambda_{i_1}}\right)^{n-3} \left(\frac{\partial}{\partial \Lambda_{i_2}}\right) \int \cdots \int D_{n-3}([i_1 - \sigma(i_1)]) D_1([i_2 - \sigma(i_2)]) \\
\times \; & \prod_{j(\neq i_1, i_2)} D_0([j - \sigma(j)]) \\
+ \; & \cdots \\
+ \; & \sum_{i_1, i_2, \cdots, i_{n-2}}^{n} \left(\frac{\partial}{\partial \Lambda_{i_1}}\right) \left(\frac{\partial}{\partial \Lambda_{i_2}}\right) \cdots \left(\frac{\partial}{\partial \Lambda_{i_{n-2}}}\right) \int \cdots \int \\
& \prod_{j=1}^{n-2} D_1([i_j - \sigma(i_j)]) \prod_{j(\neq i_1, i_2, \cdots, i_{n-2})} D_0([j - \sigma(j)]) \quad . \qquad (2.14)
\end{aligned}$$

The integrals in the equation above are with respect to z_i^*'s and we adopt a notation

$$[i] \equiv z_i^* \quad , \quad [i - j] \equiv z_i^* - z_j^* \quad . \qquad (2.15)$$

This expansion is in one to one correspondence with the following expansion of

$$\left(\sum_{i=1}^{n} x_i\right)^{n-2}$$

$$\begin{aligned}
\left(\sum_{i=1}^{n} x_i\right)^{n-2} &= \sum_{i_1=1}^{n} x_{i_1}^{n-2} + {}_{n-2}C_{n-3}\left(\Theta(n\geq 5)+\frac{1}{2}\delta_{n,4}\right)\sum_{i_1,i_2} x_{i_1}^{n-3}x_{i_2} \\
&+ {}_{n-2}C_{n-4}\left(\Theta(n\geq 7)+\frac{1}{2}\delta_{n,6}\right)\sum_{i_1,i_2} x_{i_1}^{n-4}x_{i_2}^{2} \\
&+ {}_{n-2}C_{n-4}\left(\Theta(n\geq 6)+\frac{1}{3}\delta_{n,5}\right)\sum_{i_1,i_2,i_3} x_{i_1}^{n-4}x_{i_2}x_{i_3} \\
&+ {}_{n-2}C_{n-5}\left(\Theta(n\geq 9)+\frac{1}{2}\delta_{n,8}\right)\sum_{i_1,i_2} x_{i_1}^{n-5}x_{i_2}^{3} \\
&+ {}_{n-2}C_{n-3}\ {}_{3}C_{2}\left(\Theta(n\geq 8)+\frac{1}{2}\delta_{n,7}\right)\sum_{i_1,i_2,i_3} x_{i_1}^{n-5}x_{i_2}^{2}x_{i_3} \\
&+ {}_{n-2}C_{n-5}\left(\Theta(n\geq 9)+\frac{1}{4}\delta_{n,6}\right)\sum_{i_1,i_2,i_3,i_4} x_{i_1}^{n-5}x_{i_2}x_{i_3}x_{i_4} \\
&+ \cdots \\
&+ \sum_{i_1,i_2,\cdots,i_{n-2}} x_{i_1}x_{i_2}\cdots x_{i_{n-2}} \quad . \qquad (2.16)
\end{aligned}$$

Here the summations without a parenthesis are over k different integers $i_1, i_2, \cdots i_k$, $k = 1 \sim n-2$. The number of terms appearing is equal to the number of partitions of $(n-2)$ objects into parts.

In order to put eqs. (2.13), (2.14) in a simpler form, let us introduce

$$\begin{aligned}
&\begin{pmatrix} m_1, & m_2, & \cdots, & m_n \\ i_1, & i_2, & \cdots, & i_n \end{pmatrix}_n \\
&\equiv -\sum_{\sigma\in\mathcal{S}_n} D_{m_1}([i_1-\sigma(i_1)])D_{m_2}([i_2-\sigma(i_2)])\cdots D_{m_n}([i_k-\sigma(i_n)]) \\
&= -\sum_{\sigma\in\mathcal{S}_n} \frac{1}{[i_1-\sigma(i_1)]^{m_1+1}}\ \frac{1}{[i_2-\sigma(i_2)]^{m_2+1}}\ \cdots \frac{1}{[i_n-\sigma(i_n)]^{m_n+1}} \quad . \qquad (2.17)
\end{aligned}$$

In particular,

$$\begin{pmatrix} n-2, & 0, & \cdots, & 0 \\ i_1, & i_2, & \cdots, & 0 \end{pmatrix}_n \equiv -\sum_{\sigma\in\mathcal{S}_n} D_{n-2}([i_1-\sigma(i_1)])\prod_{j(\neq i_1)} D_0([j-\sigma(j)])$$

$$\begin{pmatrix} n-3, & 1, & 0, & \cdots, & 0 \\ i_1, & i_2, & i_3, & \cdots, & 0 \end{pmatrix}_n \equiv -\sum_{\sigma\in\mathcal{S}_n} D_{n-3}([i_1-\sigma(i_1)])D_1([i_2-\sigma(i_2)]) \times \prod_{j(\neq i_1,i_2)} D_0([j-\sigma(j)])$$

$$\text{e.t.c} \tag{2.18}$$

In the appendix A, we prove that

$$\begin{pmatrix} m_1, & m_2, & \cdots, & m_n \\ i_1, & i_2, & \cdots, & i_n \end{pmatrix}_n = 0 \quad \text{if} \quad \sum_\ell m_\ell \le n-3 \quad , \tag{2.19}$$

as well as

$$\begin{pmatrix} m_1, & m_2, & \cdots, & m_k, & 0, & \cdots, & 0 \\ i_1, & i_2, & \cdots, & i_k, & i_{k+1}, & \cdots, & 0 \end{pmatrix}_n$$
$$= \sum_{\ell=1}^{k} \frac{1}{[i_\ell - i_n]^2} \begin{pmatrix} m_1, & \cdots, & m_\ell - 1, & \cdots, & m_k, & 0, & \cdots \\ i_1, & \cdots, & i_\ell, & \cdots, & i_k, & \cdots, & \cdots \end{pmatrix}_{n-1}$$
$$\text{if} \quad \sum_\ell m_\ell = n-2 \quad . \tag{2.20}$$

In particular,

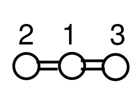
$$\begin{pmatrix} n-2, & 0, & \cdots \\ i_1, & \cdots, & \cdots \end{pmatrix}_n = \frac{1}{[i_1-i_n]^2} \begin{pmatrix} n-3, & 0, & \cdots \\ i_1, & \cdots, & \cdots \end{pmatrix}_{n-1} = \frac{1}{\prod_{j(\neq i_1)}^{n} [i_1 - j]^2} \tag{2.21}$$

and

$$\begin{pmatrix} n-3, & 1, & \cdots \\ i_1, & i_2 & \cdots \end{pmatrix}_n = \frac{1}{[i_1-i_n]^2} \begin{pmatrix} n-4, & 1, & \cdots \\ i_1, & i_2, & \cdots \end{pmatrix}_{n-1} + \frac{1}{[i_2-i_n]^2} \begin{pmatrix} n-3, & 0, & \cdots \\ i_1, & i_2, & \cdots \end{pmatrix}_{n-1} \quad . \tag{2.22}$$

Let us introduce graphs in which the factor $1/[i-j]^2$ is represented by a double line linking circle i and circle j to handle the quantities defined by eq. (2.17) more easily. For example, for n=3,

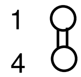
$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 3 \end{pmatrix}_3 = \frac{1}{[z_1^* - z_2^*]^2} \frac{1}{[z_1^* - z_3^*]^2} \equiv$$

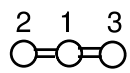


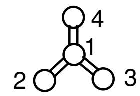
$$. \tag{2.23}$$

Using the recursin relation eq. (2.20) and eq. (2.19) we have, for n=4,

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{pmatrix}_4 = \frac{1}{[z_1^* - z_4^*]^2} \begin{pmatrix} 2-1 & 0 & 0 \\ 1 & 2 & 3 \end{pmatrix}_3$$

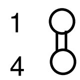
$=$ 

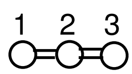
$\times$ 

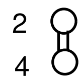
$=$ 

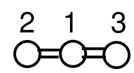
, (2.24)

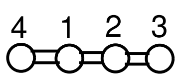
$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{pmatrix}_4 = \frac{1}{[z_1^* - z_4^*]^2} \begin{pmatrix} 1-1 & 1 & 0 \\ 1 & 2 & 3 \end{pmatrix}_3 + \frac{1}{[z_2^* - z_4^*]^2} \begin{pmatrix} 1 & 1-1 & 0 \\ 1 & 2 & 3 \end{pmatrix}_3$$

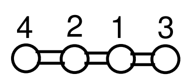
$=$ 

$\times$ 

$+$ 

$\times$ 

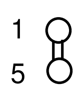
$=$ 

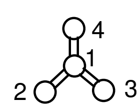
$+$ 

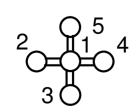
, (2.25)

and, for n=5,

$$\begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}_5 = \frac{1}{[z_1^* - z_5^*]^2} \begin{pmatrix} 3-1 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{pmatrix}_4$$

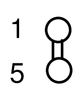
$=$ 

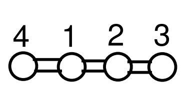
$\times$ 

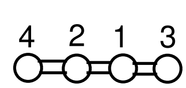
$=$ 

, (2.26)

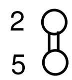
$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}_5 = \frac{1}{[z_1^* - z_5^*]^2} \begin{pmatrix} 2-1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{pmatrix}_4 + \frac{1}{[z_2^* - z_5^*]^2} \begin{pmatrix} 2 & 1-1 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{pmatrix}_4$$

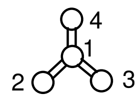
$=$ 

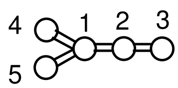
$\times \Big\{$ 

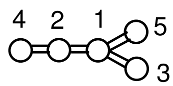
$+$ 

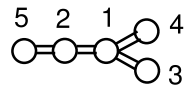
$\Big\}$

$+$ 

$\times$ 

$=$ 

$+$ 

$+$ 

$.$ (2.27)

From the examples above, it is clear that the graphs for the general case can be written down quite easily.

In terms of the quantities defined by eq. (2.17), we obtain a formula for the n-point resolvent :

$$\left(\frac{N}{\Lambda}\right)^{n-2} << \prod_{i=1}^{n} tr \frac{1}{p_i - \hat{M}} >>_{N,conn} = \prod_{i=1}^{n} \left(-\frac{\partial}{\partial(a\zeta_i)}\right) \frac{-1}{(n-2)!} \tilde{\Delta}_n|_{\Lambda_i=\Lambda} \quad , \qquad (2.28)$$

where

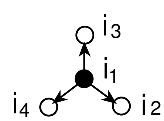
$$\begin{aligned}
&-\frac{1}{(n-2)!}\tilde{\Delta}_n(z_1^*, \cdots . z_n^*) \\
&\qquad = \sum_{i_1}^{n} \Big(\frac{\partial}{\partial \Lambda_{i_1}}\Big)^{n-2} \int \cdots \int \begin{pmatrix} n-2, & 0, & \cdots, & 0 \\ i_1, & i_2, & \cdots, & i_n \end{pmatrix}_n \\
&\qquad + \sum_{m_1+m_2=n-2} \sum_{(i_1,i_2)} \Big(\frac{\partial}{\partial \Lambda_{i_1}}\Big)^{m_1} \Big(\frac{\partial}{\partial \Lambda_{i_2}}\Big)^{m_2} \int \cdots \int \begin{pmatrix} m_1, & m_2, & 0, & \cdots, & 0 \\ i_1, & i_2, & i_3, & \cdots, & i_n \end{pmatrix}_n \\
&\qquad + \sum_{\substack{m_1+m_2+m_3 \\ =n-2}} \sum_{(i_1,i_2,i_3)} \Big(\frac{\partial}{\partial \Lambda_{i_1}}\Big)^{m_1} \Big(\frac{\partial}{\partial \Lambda_{i_2}}\Big)^{m_2} \Big(\frac{\partial}{\partial \Lambda_{i_3}}\Big)^{m_3} \int \cdots \int \begin{pmatrix} m_1, & m_2, & m_3, & 0, & \cdots, & 0 \\ i_1, & i_2, & i_3, & i_4, & \cdots, & i_n \end{pmatrix}_n \\
&\qquad + \quad \cdots \\
&\qquad + \sum_{(i_1,\cdots,i_{n-2})} \Big(\frac{\partial}{\partial \Lambda_{i_1}}\Big) \cdots \Big(\frac{\partial}{\partial \Lambda_{i_{n-2}}}\Big) \int \cdots \int \begin{pmatrix} 1, & \cdots, & 1, & 0, & 0 \\ i_1, & \cdots, & i_{n-2}, & i_{n-1}, & i_n \end{pmatrix}_n \quad . \qquad (2.29)
\end{aligned}$$

Here $m_\ell \geq 1$ and the summation $(i_1, \cdots, i_k)$ denotes a set of k unequal integers from $1, 2, \cdots, n$ and $i_{k+1}, \cdots, i_n$ in the array represents the remaining integers. Eqs. (2.28), (2.29) are a part of our main results.

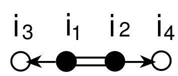
It is straightforward to write down the correlator for lower n explicitly;

$$
\begin{aligned}
\frac{\partial}{\partial\Lambda}\frac{-1}{(0)!}\tilde{\Delta}_2 &= \sum_{i=1}^{2}\frac{\partial z_i^*}{\partial\Lambda_i}\frac{1}{\prod\limits_{j(\neq i)}^{2}[i-j]} \quad , \\
\frac{-1}{(1)!}\tilde{\Delta}_3 &= \sum_{i=1}^{3}\frac{\partial z_i^*}{\partial\Lambda_i}\frac{1}{\prod\limits_{j(\neq i)}^{3}[i-j]} \quad .
\end{aligned}
\tag{2.30}
$$

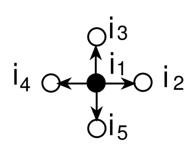
For $n = 4, 5$, the correlator can be written more compactly using graphs as introduced below:

$$\frac{-1}{2!}\tilde{\Delta}_4 = \sum \frac{\partial}{\partial\Lambda_{i_1}}\Big\{$$


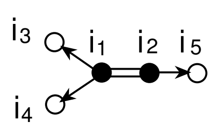
$$\Big\}$$

$$+ \sum \Big\{$$


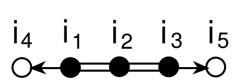
$$\Big\} \quad , \tag{2.31}$$

$$\frac{-1}{3!}\tilde{\Delta}_5 = \sum \left(\frac{\partial}{\partial\Lambda_{i_1}}\right)^2\Big\{$$


$$\Big\}$$

$$+ \sum \left(\frac{\partial}{\partial\Lambda_{i_1}}\right)\Big\{$$


$$\Big\}$$

$$+ \sum \Big\{$$


$$\Big\} \quad . \tag{2.32}$$

In these figures a double line linking circle i and circle j, a single line having an arrow from circle i to circle j and a solid circle i represent $1/[i-j]^2$, $1/[i-j]$ and $\frac{\partial z_i^*}{\partial\Lambda_i}$ respectively. The summations are over all possible graphs that have the same

topology specified. Each graph appears just for once in the summation. Note that the links to the external circles are not double lines but the single ones with arrows and that the internal circles are solid circles.

For general n, $-\frac{1}{(n-2)!}\tilde{\Delta}_n$ is expressed in the same way. The rule is as follows. First, we consider all possible graphs which have n circles and $n-1$ links in the same way as in the case $n=5$. Second, if the internal solid circle i has ℓ_i links in each graph, the graph is operated by $\prod_i \left(\frac{\partial}{\partial\Lambda_i}\right)^{\ell_{i-2}}$. Then the summation over all graphs gives the expression for $-\frac{1}{(n-2)!}\tilde{\Delta}_n$.

Isolating the term which comes with the highest number of total derivatives of the bare cosmological constant Λ (as opposed to Λ_i) by

$$
\begin{aligned}
\sum_{i_1}\left(\frac{\partial}{\partial\Lambda_{i_1}}\right)^{n-2} &= \left(\frac{\partial}{\partial\Lambda}\right)^{n-3}\sum_{i_1=1}^{n}\frac{\partial}{\partial\Lambda_{i_1}} \\
&- \sum_{i_1=1}^{n}\left[\sum_{k=0}^{n-4}\left(\frac{\partial}{\partial\Lambda_{i_1}}\right)^{n-k-3}\left(\sum_{\ell=1}^{n}\frac{\partial}{\partial\Lambda_\ell}\right)^{k}\right]\sum_{j(\neq i_1)}\frac{\partial}{\partial\Lambda_j} \quad ,
\end{aligned}
$$

we obtain another expression for $\frac{-1}{(n-2)!}\tilde{\Delta}_n$:

$$
\begin{aligned}
&\frac{-1}{(n-2)!}\tilde{\Delta}_n = \left(\frac{\partial}{\partial\Lambda}\right)^{n-3}\sum_{i_1}^{n}\left(\frac{\partial}{\partial\Lambda_{i_1}}\right)\int\cdots\int\begin{pmatrix} n-2, & 0, & \cdots \\ i_1, & \cdots, & \cdots \end{pmatrix} \\
&+ \sum_{i_1,i_2}^{n}\left(\frac{\partial}{\partial\Lambda_{i_1}}\right)^{n-3}\left(\frac{\partial}{\partial\Lambda_{i_2}}\right)\int\cdots\int\left\{\left(\Theta(n\geq 5)+\frac{1}{2}\delta_{n,4}\right)\begin{pmatrix} n-3, & 1, & 0, & \cdots \\ i_1, & i_2, & \cdots, & \cdots \end{pmatrix}\right. \\
&\left. -(n-3)\begin{pmatrix} n-2, & 0, & \cdots \\ i_1, & i_2, & \cdots \end{pmatrix}\right\} \\
&+ \sum_{i_1,i_2}^{n}\left(\frac{\partial}{\partial\Lambda_{i_1}}\right)^{n-4}\left(\frac{\partial}{\partial\Lambda_{i_2}}\right)^{2}\int\cdots\int\left\{\left(\Theta(n\geq 7)+\frac{1}{2}\delta_{n,6}\right)\begin{pmatrix} n-4, & 2, & 0 & \cdots \\ i_1, & i_2, & \cdots & \cdots \end{pmatrix}\right. \\
&\left. -\left(\sum_{k=0}^{n-4}{}_kC_1\right)\begin{pmatrix} n-2, & 0, & \cdots \\ i_1, & i_2, & \cdots \end{pmatrix}\right\} \\
&+ \sum_{i_1,(i_2,i_3)}^{n}\left(\frac{\partial}{\partial\Lambda_{i_1}}\right)^{n-4}\left(\frac{\partial}{\partial\Lambda_{i_2}}\right)\left(\frac{\partial}{\partial\Lambda_{i_3}}\right)\int\cdots\int\left\{\left(\Theta(n\geq 6)+\frac{1}{3}\delta_{n,5}\right)\times\right. \\
&\left.\begin{pmatrix} n-4, & 1, & 1 & 0 \\ i_1, & i_2 & i_3 & \end{pmatrix} - \left(\sum_{k=0}^{n-4}{}_kC_1\right)\begin{pmatrix} n-2, & 0, & 0, & \cdots \\ i_1, & i_2 & i_3, & \cdots \end{pmatrix}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{i_1,i_2}^{n}\left(\frac{\partial}{\partial\Lambda_{i_1}}\right)^{n-5}\left(\frac{\partial}{\partial\Lambda_{i_2}}\right)^{3}\int\cdots\int\left\{\left(\Theta(n\geq 9)+\frac{1}{2}\delta_{n,8}\right)\begin{pmatrix} n-5, & 3, & \cdots & \\ i_1, & i_2 & & \end{pmatrix}\right. \\
&\quad \left. -\left(\sum_{k=0}^{n-4} {}_kC_2\right)\begin{pmatrix} n-2, & 0, & \cdots & \cdots \\ i_1, & i_2, & \cdots & \cdots \end{pmatrix}\right\} \\
&+ \sum_{i_1,i_2,i_3}^{n}\left(\frac{\partial}{\partial\Lambda_{i_1}}\right)^{n-5}\left(\frac{\partial}{\partial\Lambda_{i_2}}\right)^{2}\left(\frac{\partial}{\partial\Lambda_{i_3}}\right)\int\cdots\int\left\{\left(\Theta(n\geq 8)+\frac{1}{2}\delta_{n,7}\right)\times\right. \\
&\quad \left.\begin{pmatrix} n-5, & 2, & 1, & \cdots \\ i_1, & i_2, & i_3, & \cdots \end{pmatrix} - \left(\sum_{k=0}^{n-4} {}_kC_2\right)\begin{pmatrix} n-2, & 0, & 0 & \cdots \\ i_1, & i_2, & i_3, & \cdots \end{pmatrix}\right\} \\
&+ \sum_{i_1,(i_2,i_3,i_4)}^{n}\left(\frac{\partial}{\partial\Lambda_{i_1}}\right)^{n-5}\left(\frac{\partial}{\partial\Lambda_{i_2}}\right)\left(\frac{\partial}{\partial\Lambda_{i_3}}\right)\left(\frac{\partial}{\partial\Lambda_{i_4}}\right)\int\cdots\int\left\{\left(\Theta(n\geq 7)+\frac{1}{4}\delta_{n,6}\right)\times\right. \\
&\quad \left.\begin{pmatrix} n-3, & 1, & 1, & 1, & \cdots \\ i_1, & i_2, & i_3, & i_4, & \cdots \end{pmatrix} - \left(\sum_{k=0}^{n-4} {}_kC_2\right)\begin{pmatrix} n-2, & 0, & 0, & 0, & \cdots \\ i_1, & i_2, & i_3, & \cdots & \end{pmatrix}\right\} \\
&+ \cdots \\
&+ \sum_{(i_1,i_2,\cdots,i_{n-2})}^{n}\left(\frac{\partial}{\partial\Lambda_{i_1}}\right)\cdots\left(\frac{\partial}{\partial\Lambda_{i_{n-2}}}\right)\int\cdots\int\left\{\begin{pmatrix} 1, & 1, & \cdots & 1, & 0, & \cdots \\ i_1, & i_2, & \cdots & i_{n-2}, & \cdots & \cdots \end{pmatrix}\right. \\
&- \left.\begin{pmatrix} n-2, & 0, & \cdots & 0, & 0, & 0 \\ i_1, & i_2, & \cdots & i_{n-2}, & \cdots & \cdots \end{pmatrix} - \cdots - \begin{pmatrix} 0, & 0, & \cdots & n-2, & 0 & 0 \\ i_1, & i_2, & \cdots & i_{n-2}, & \cdots & \cdots \end{pmatrix}\right\} .
\end{aligned}
\tag{2.33}
$$

Here the summations with a parenthesis $(i_1, i_2, \cdots, i_k)$ are over k unequal indistinguished indices. Using this formula and the partial fraction, one can put the expression into a form which does not contain a direct link among solid circles. This is sometimes useful in the manipulation we carry out later.

The following parametrization of z_i^* [6] is important in the next section:

$$z_i^* = \exp(2\eta\cosh\theta_i) \quad . \tag{2.34}$$

This parametrization is understood together with

$$p_i - p_i^* = [\hat{M}](z_i;\Lambda_i) - [\hat{M}]^* = aM\cosh m\theta_i \quad , \quad \eta = (aM/2)^{1/m} \tag{2.35}$$

and

$$\Lambda - \Lambda_* = -(m+1)\eta^{2m} = -(m+1)a^2t \ , \quad M = (2t)^{1/2} \quad . \tag{2.36}$$

We obtain

$$\frac{\partial\theta_i}{\partial\Lambda}\,|_{\zeta_i}= -\frac{1}{\eta}\frac{\partial\eta}{\partial\Lambda}\frac{\cosh m\theta_i}{\sinh m\theta_i} \quad , \quad \frac{\partial z_i^*}{\partial\Lambda_i} = 2\left(\frac{\partial\eta}{\partial\Lambda}\right)\frac{\sinh(m-1)\theta_i}{\sinh m\theta_i} \quad . \tag{2.37}$$

The origin of the parametrization eq. (2.34) comes from the planar solution to the Heisenberg algebra [13] in [6]. In fact, eq. (2.35) represents the solution at the $(m+1, m)$ critical point. (p_i^* and $[\hat{M}]^*$ represent the critical value to p_i and to $[\hat{M}]$ respectively.)

III. Fusion Rules, Crossing Symmetry and Polygons Associated

We now discuss the term we have isolated in eq. (2.33), the expression for the n-point resolvent, namely the one which comes with the highest number of total derivatives with respect to Λ and is, therefore, familiar from the case of pure two-dimensional gravity. We will exhibit striking properties with this term, using the parametrization noted in eq. (2.34). Let us define

$$\begin{aligned} P_n(\theta_1,\theta_2,\cdots\theta_n) &\equiv \sum_{i_1}^{n}\left(\frac{\partial}{\partial\Lambda_{i_1}}\right)\int\cdots\int\begin{pmatrix} n-2, & 0, & \cdots, & 0 \\ i_1, & i_2, & \cdots, & i_n \end{pmatrix} \\ &= \sum_{i=1}^{n}\frac{\partial z_i^*}{\partial\Lambda_i}\frac{1}{\prod\limits_{j(\neq i)}^{n}[i-j]} \quad . \end{aligned} \tag{3.1}$$

A key manipulation we will use is the partial fraction

$$\frac{1}{[i-j][i-k]} = \frac{1}{[i-k][k-j]} + \frac{1}{[i-j][j-k]} \quad . \tag{3.2}$$

One can associate a line from i to j with $\frac{1}{[i-j]}$. The following identity is responsible for expressing P_n as a sum of the product of n factors each of which depends only on θ_i:

$$\begin{aligned} I_k(\alpha,\beta;m) &\equiv \frac{1}{\cosh\alpha-\cosh\beta}\left(\frac{\sinh(m-k)\alpha}{\sinh m\alpha}-\frac{\sinh(m-k)\beta}{\sinh m\beta}\right) \\ &= -2\sum_{j=1}^{m-k}\sum_{i=1}^{k}\frac{\sinh(m-j-i+1)\alpha}{\sinh m\alpha}\,\frac{\sinh(m-j-k+i)\beta}{\sinh m\beta} \end{aligned} \tag{3.3}$$

Let us first work out the cases $n = 2, 3, 4$ to get a feeling. For $n = 2$,

$$
\begin{aligned}
P_2(\theta_1, \theta_2) &= \frac{\frac{\partial z_1^*}{\partial \Lambda_1} - \frac{\partial z_2^*}{\partial \Lambda_2}}{[1-2]} = 2\left(\frac{\partial \eta}{\partial \Lambda}\right)\frac{1}{2\eta} I_{k_1=1}(\theta_1, \theta_2; m) \\
&= 2\left(\frac{\partial \eta}{\partial \Lambda}\right)\left(\frac{-1}{\eta}\right)\sum_{j=1}^{m-k_1}\sum_{i_1=1}^{k_1}\frac{\sinh(m-j_1-i_1+1)\theta_1}{\sinh m\theta_1}\frac{\sinh(m-j_1+i_1-k_1)\theta_2}{\sinh m\theta_2} \\
&= 2\left(\frac{\partial \eta}{\partial \Lambda}\right)\left(\frac{-1}{\eta}\right)\sum_{j_1=1}^{m-1}\frac{\sinh(m-j_1)\theta_1}{\sinh m\theta_1}\frac{\sinh(m-j_1)\theta_2}{\sinh m\theta_2} \quad , \qquad (3.4)
\end{aligned}
$$

where $i_1 = k_1 = 1$. For $n = 3$, we use eq. (3.2) for the term containing $\frac{\partial z_2^*}{\partial \Lambda_2}$ to create a link $[1-3]$, which is originally absent. This relates P_3 to P_2. We find

$$
\begin{aligned}
&P_3(\theta_1, \theta_2, \theta_3) \\
&= 2\left(\frac{\partial \eta}{\partial \Lambda}\right)\frac{I_1(\theta_2, \theta_1; m) - I_1(\theta_2, \theta_3; m)}{[1-3]} \\
&= 2\left(\frac{\partial \eta}{\partial \Lambda}\right)\left(\frac{-1}{\eta}\right)^2\left(\sum_{j_1=1}^{m-k_1}\sum_{i_1=1}^{k_1=1}\right)\left(\sum_{j_2=1}^{m-k_2}\sum_{i_2=1}^{k_2}\right) \qquad (3.5) \\
&\quad \frac{\sinh(m-j_2-i_2+1)\theta_1}{\sinh m\theta_1}\frac{\sinh(m-j_1+i_1-k_1)\theta_2}{\sinh m\theta_2}\frac{\sinh(m-j_2+i_2-k_2)\theta_3}{\sinh m\theta_3} \; .
\end{aligned}
$$

Here $k_2 = j_1 + i_1 - 1 = j_1$.

This can be repeated for arbitrary n. In the case $n = 4$, we use the partial fraction for the two terms containing $\frac{\partial z_2^*}{\partial \Lambda_2}$ and $\frac{\partial z_3^*}{\partial \Lambda_3}$ to create a link $[1-4]$, which is originally absent. This enables us to relate the case $n = 4$ to the case $n = 3$. In general, P_n is related to P_{n-1} by using the partial fraction for the terms containing $\frac{\partial z_2^*}{\partial \Lambda_2} \sim \frac{\partial z_{n-1}^*}{\partial \Lambda_{n-1}}$ to create a link $[1-n]$. We obtain

$$
\begin{aligned}
P_n(\theta_1, \theta_2, \cdots \theta_n) &= -\left(\frac{\partial \eta^2}{\partial \Lambda}\right)\left(\frac{-1}{\eta}\right)^n\left(\prod_{\ell=1}^{n-1}\sum_{j_\ell=1}^{m-k_\ell}\sum_{i_\ell=1}^{k_\ell}\right) \\
&\left(\prod_{\ell'=1}^{n-1}\frac{\sinh(m-j_{\ell'}+i_{\ell'}-k_{\ell'})\theta_{\ell'+1}}{\sinh m\theta_{\ell'+1}}\right)\frac{\sinh(m-j_{n-1}-i_{n-1}+1)\theta_1}{\sinh m\theta_1} \quad , \qquad (3.6)
\end{aligned}
$$

where $k_\ell = j_{\ell-1} + i_{\ell-1} - 1$, for $\ell = 2, 3, \cdots, (n-1)$. Eq. (3.6) expresses the $P_n(\theta_1, \theta_2, \cdots \theta_n)$ as a sum of the n-products of the factor $\dfrac{\sinh(m-k)\theta_i}{\sinh m\theta_i}$. Owing to this property, one can perform the inverse Laplace transform immediately, which we will carry out at eq. (3.22).

Let us now discuss the restrictions on the summations of $2n-3$ integers j_1, i_2, j_2, $\cdots i_{n-1}, j_{n-1}$ in eq. (3.6). We write these as a set:

$$\begin{aligned}
&\mathcal{F}_n(j_1, i_2, j_2 \cdots i_{n-1}, j_{n-1}) \\
\equiv\ & \{(j_1, i_2, j_2, \cdots i_{n-1}, j_{n-1}) \mid 1 \le i_\ell \le k_\ell,\ 1 \le j_\ell \le m - k_\ell,\ \text{for } \ell = 1, 2, \cdots n-1\} \\
=\ & \mathcal{F}_2(i_1 = 1, j_1; k_1 = 1) \prod_{\ell=2}^{n-1} \cap \mathcal{F}_2(i_\ell, j_\ell; k_\ell) \ ,
\end{aligned} \tag{3.7}$$

where

$$\mathcal{F}_2(i_\ell, j_\ell; k_\ell) \equiv \{(i_\ell, j_\ell) \mid 1 \le i_\ell \le k_\ell,\ 1 \le j_\ell \le m - k_\ell,\ \ \text{with } k_\ell \ \ \text{fixed}\} \ . \tag{3.8}$$

We will show that these restrictions on the sums are in fact in one-to-one correspondence with the fusion rules of the unitary minimal models for the diagonal primaries. Let us begin with the case $n = 3$. Define

$$\begin{aligned}
&p_1 \equiv j_1 + k_1 - i_1 \ , \quad p_2 \equiv j_2 + k_2 - i_2 \ , \quad q_3 \equiv j_2 + i_2 - 1 \ , \\
&a_{12} \equiv p_1 - 1 \ , \qquad\quad a_{23} \equiv p_2 - 1 \ , \qquad\quad a_{31} \equiv q_3 - 1 \ ,
\end{aligned} \tag{3.9}$$

The inequalities on i_2, j_2 are found to be equivalent to the following four inequalities:

$$\begin{aligned}
a_{12} + a_{23} - a_{31} &= 2(k_2 - i_2) \ge 0 \quad . \\
a_{12} - a_{23} + a_{31} &= 2(i_2 - 1) \ge 0 \quad . \\
-a_{12} + a_{23} + a_{31} &= 2(j_2 - 1) \ge 0 \\
a_{12} + a_{23} + a_{31} &= 2(j_2 + k_2 - 2) \le 2(m-2) \ .
\end{aligned} \tag{3.10}$$

From the third and the fourth equation of eq. (3.10), the inequality $a_{12} \le m - 2$ follows, which is a condition for $\mathcal{F}_2(i_1 = 1, j_1; k_1 = 1)$. Defining a set

$$\begin{aligned}
\mathcal{D}_3(a_1, a_2, a_3) \ \equiv\ & \{(a_1, a_2, a_3) \mid \sum_{i(\ne j)}^{3} a_i - a_j \ge 0 \ \text{ for } \ i = 1 \sim 3 \quad , \\
& \sum_{i=1}^{3} a_i = \text{even} \le 2(m-2)\} \ ,
\end{aligned} \tag{3.11}$$

we state eq. (3.10) as

$$\mathcal{F}_3(j_1, i_2, j_2) = \mathcal{D}_3(a_{12}, a_{23}, a_{31}) \quad . \tag{3.12}$$

We also write

$$\mathcal{F}_2(j_1) \equiv \mathcal{F}_2(i_1 = 1, j_1; k_1 = 1) \equiv \mathcal{D}_2(a_{12}) \quad . \tag{3.13}$$

for the case $n = 2$.

Eq. (3.11) is nothing but the condition that a triangle be formed which is made out of a_1, a_2 and a_3 and whose circumference is less than or equal to $2(m-2)$. It is also the selection rule for the three point function of the diagonal primaries in m-th minimal unitary conformal field theory [7]. In fact, the fusion rules for diagonal primary fields read as

$$\langle \phi_{ii}\ \phi_{jj}\ \phi_{kk} \rangle \neq 0 \quad , \tag{3.14}$$

if and only if $i+j \geq k+1$ and two other permutations and $i+j+k$ $(=\text{odd}) \leq 2m-1$ hold. This set of rules is nothing but $\mathcal{D}_3(i-1, j-1, k-1)$.

For the case $n = 4$, introduce $p_3 \equiv j_3 + k_3 - i_3$, $a_{34} \equiv p_3 - 1$, $q_4 \equiv j_3 + i_3 - 1$, $a_{41} \equiv q_4 - 1$. We find

$$\mathcal{F}_2(i_3, j_3; k_3) = \mathcal{D}_3(a_{31}, a_{34}, a_{41}) \tag{3.15}$$

The restrictions on the sum in the case $n = 4$ can be understood as gluing the two triangels:

$$\begin{aligned} \mathcal{F}_4(j_1, i_2, j_2, i_3, j_3) &= \mathcal{D}_3(a_{12}, a_{23}, a_{31}) \cap \mathcal{D}_3(a_{34}, a_{41}, a_{31}) \\ &\equiv \mathcal{D}_4(a_{12}, a_{23}, a_{34}, a_{41}; a_{31}) \quad . \end{aligned} \tag{3.16}$$

The allowed integers on a_{31} are naturally interpreted as permissible quantum numbers flowing through an intermediate channel. As one can imagine, eq. (3.16) is not the only way to represent the restriction: one can also represent it as

$$\begin{aligned} &\mathcal{F}_4(j_1, i_2, j_2, i_3, j_3) = \mathcal{D}_3(a_{12}, a_{24}, a_{41}) \cap \mathcal{D}_3(a_{23}, a_{34}, a_{24}) \\ &\equiv \mathcal{D}_4(a_{12}, a_{23}, a_{34}, a_{41}; a_{24}) \quad , \end{aligned} \tag{3.17}$$

which embodies the crossing symmetric property of the amplitude.

The restrictions in the general case n are understood as attaching a triangle to the case $(n-1)$. To see this, define

$$\begin{array}{ll} p_\ell = j_\ell + k_\ell - i_\ell \ , & q_\ell = j_{\ell-1} + i_{\ell-1} - 1 \ , \\ a_{\ell,1} = q_\ell - 1 \ , & a_{\ell,\ell+1} = p_\ell - 1 \ , \end{array} \quad \text{for} \quad \ell = 1, 2, \cdots \text{n} \quad . \tag{3.18}$$

Using $1 \le i_{n-1} \le k_{n-1}, \quad 1 \le j_{n-1} \le m - k_{n-1}$, we derive

$$\begin{aligned} a_{n-1,n} + a_{n,1} - a_{n-1,1} &= 2(j_{n-1} - 1) \ge 0 \ , \\ a_{n-1,n} - a_{n,1} + a_{n-1,1} &= 2(k_{n-1} - i_{n-1}) \ge 0 \ , \\ -a_{n-1,n} + a_{n,1} + a_{n-1,1} &= 2(i_{n-1} - 1) \ge 0 \ , \\ a_{n-1,n} + a_{n,1} + a_{n-1,1} &= 2(j_{n-1} + k_{n-1} - 2) \le 2(m-2) \ . \end{aligned} \tag{3.19}$$

The restriction on i_{n-1} and j_{n-1} are, therefore, $\mathcal{D}_3(a_{n-1,n}, a_{n,1}, a_{n-1,1})$, which is what we wanted to see. All in all, we find

$$\begin{aligned} & \mathcal{F}_n(j_1, i_2, j_2, \cdots i_{n-1}, j_{n-1}) \\ = \; & \mathcal{D}_3(a_{n-1,n}, a_{n,1}, a_{n-1,1}) \cap \mathcal{F}_{n-1}(j_1, i_2, j_2, \cdots i_{n-2}, j_{n-2}) \\ = \; & \mathcal{D}_3(a_{n-1,n}, a_{n,1}, a_{n-1,1}) \cap \mathcal{D}_{n-1}(a_{1,2}, a_{2,3}, \cdots a_{n-2,n-1}, a_{n-1,1}; a_{3,1}, a_{4,1}, \cdots a_{n-2,1}) \\ \equiv \; & \mathcal{D}_n(a_{1,2}, a_{2,3}, \cdots a_{n-1,n}, a_{n,1}; a_{3,1}, a_{4,1}, \cdots a_{n-1,1}) \end{aligned} \tag{3.20}$$

From now on, a shortened notation $\mathcal{D}_n(a_{1,2}, a_{2,3}, \cdots a_{n-1,n}, a_{n,1})$ is understood to represent $\mathcal{D}_n(a_{1,2}, a_{2,3}, \cdots a_{n-1,n}, a_{n,1}; a_{3,1}, a_{4,1}, \cdots a_{n-1,1})$.

Putting eq (3.6) and eq. (3.20) together, we obtain a formula

$$\begin{aligned} & P_n(\theta_1, \theta_2, \cdots \theta_n) \\ = \; & - \left(\frac{\partial \eta^2}{\partial \Lambda} \right) \left(\frac{-1}{\eta} \right)^n \sum_{\mathcal{D}_n} \left(\prod_{j=2}^{n} \frac{\sinh(m - k_j - 1)\theta_j}{\sinh m\theta_j} \right) \frac{\sinh(m - k_1 - 1)\theta_1}{\sinh m\theta_1} \quad , \end{aligned} \tag{3.21}$$

where $\mathcal{D}_n$ means $\mathcal{D}_n(k_1 - 1, \cdots, k_n - 1)$. Once again, the fact that the different divisions of $\mathcal{D}_n$ into $n-2$ triangles are embodied by this single expression is precisely the statement of the old duality.

The object $P_n(\theta_1, \theta_2, \cdots \theta_n)$ is equipped with θ_j and k_j for $j = 1, 2, \cdots, n$ and any $\mathcal{D}_3(k_1 - 1, k_2 - 1, k_3 - 1)$ obeys the rule of the triangle specified above. It is, therefore, natural to visualize this as a vertex which connects n external legs corresponding to n loops. The vertex can be regarded as a dual graph of an n-gon that corresponds $\mathcal{D}_n(k_1 - 1, \cdots, k_n - 1)$.

Using the formula (1.4), we perform the inverse Laplace transform with respect to ζ_i $(i = 1, \sim n)$. We obtain

$$\begin{aligned}&\left(\prod_{j=1}^{n} \mathcal{L}_j^{-1}\right)\left(\frac{\Lambda}{N}\right)^{n-2}\left(\frac{\partial}{\partial\Lambda}\right)^{n-3}\prod_{i=1}^{n}\left(-\frac{\partial}{\partial(a\zeta_i)}\right)P_n(\theta_1,\theta_2,\cdots,\theta_n)\\ &= (-)^{n+1}\left(\frac{\Lambda}{N}\right)^{n-2}\left(\frac{\partial}{\partial\Lambda}\right)^{n-3}\left[\frac{\partial\eta^2}{\partial\Lambda}\left(\frac{1}{a\eta}\right)^{n}\sum_{\mathcal{D}_n}\prod_{j=1}^{n}\frac{M\ell_j}{\pi}\underline{K}_{1-k_j/m}(M\ell_j)\right], \qquad (3.22)\end{aligned}$$

where $\mathcal{L}_j^{-1}$ denotes the inverse Laplace transform with respect to ζ_j. Expressing this by a and t we obtain

$$\begin{aligned}&\mathcal{A}_n^{fusion}(\ell_1,\ell_2,\cdots\ell_n)\\ &= -\frac{1}{m}\left(\frac{1}{m+1}\right)^{n-2}\left(\frac{\partial}{\partial t}\right)^{n-3}\left[t^{-1-\frac{(n-2)}{2m}}\sum_{\mathcal{D}_n}\prod_{j=1}^{n}\frac{M\ell_j}{\pi}\underline{K}_{1-k_j/m}\left(M\ell_j\right)\right]. \qquad (3.23)\end{aligned}$$

This is the answer quoted in the introduction.

It is straightforward to look at the small length behavior of eq.(3.23). This was done in [9] in the case $n = 3$, using the formula

$$K_\nu(x) = \frac{\pi}{2\sin(\nu\pi)}\left[\frac{1}{x^\nu}\left\{\frac{2^\nu}{\Gamma(1-\nu)} + O(x^2)\right\} + x^\nu\left\{-\frac{1}{2^\nu\Gamma(1+\nu)} + O(x^2)\right\}\right]. \qquad (3.24)$$

The agreement with the approach from the generalized Kdv flows [14] (See also [15].) has been given. We will not dwell on this point further.

IV. Residual Interactions

Our formula in the last section tells how the higher order operators (gravitational descendants) in addition to the dressed primaries included in the form of the loop length are constrained to obey the selection rules of CFT. The two-matrix model realizing the unitary minimal series coupled to gravity as the continuum limit of the $(m+1, m)$ symmetric critical point knows the fusion rules and the duality symmetry in the form of the loop operators. The term we have dealt with in the last section for general n is, however, supplemented with an increasing number of other terms with n $(n \geq 4)$. The existence of such terms itself implies that the knowledge we obtain from the two and the three point functions is not sufficient to determine the full

amplitude for $n \geq 4$. This coincides with the notion of contact interactions familiar from the field theory of derivative couplings as well as in (super) string theory [16]. The counterpart of our approach to this phenomenon in the continuum framework is presumably related to the discusson on the boundary of moduli space.

In what follows, we show how to perform the inverse Laplace transformation of the resolvents to get loop amplitudes in terms of loop lengths in the case of $n = 4$, 5. It is necessary to put

$$\tilde{\Delta}_n(\theta_1, \cdots, \theta_n) \equiv \tilde{\Delta}_n(z_1^*, \cdots, z_n^*)|_{\Lambda_i=\Lambda} \tag{4.1}$$

in a manageable form to the inverse Laplace transform. Let us recall that $P_n(\theta_1, \cdots, \theta_n)$ can be inverse Laplace transformed immediately. If $\tilde{\Delta}_n(\theta_1, \cdots, \theta_n)$ is expressed as a polynomial of $P_j(\theta_1, \cdots, \theta_n)$ and their derivatives with respect to Λ, the inverse Laplace transform can be done immediately. Let us pursue this possibility. We also make use of the fact that when one of the loops shrinks and the loop length goes to zero, the n-loop amplitude must become proportional to the derivative of the $(n-1)$-loop amplitude with respect to the cosmological constant. We represent this fact by

$$\mathcal{A}_n(\ell_1, \cdots, \ell_n) \to \quad \propto \frac{\partial}{\partial t}\mathcal{A}_{n-1}(\ell_1, \cdots, \ell_{n-1}) \quad . \tag{4.2}$$

The inverse Laplace transformation of $P_n(\theta_1, \cdots, \theta_n)$ is

$$\mathcal{L}^{-1}\left[P_n(\theta_1, \cdots, \theta_n)\right] = -\Big(\frac{\partial \eta^2}{\partial \Lambda}\Big)\Big(\frac{-1}{\eta}\Big)^n\Big(\frac{M}{\pi}\Big)^n \sum_{\mathcal{D}_n}\Big[\prod_i \underline{K}_{1-k_i/m}(M\ell_i)\Big] \tag{4.3}$$

and in the limit $M\ell \to 0$ we have

$$\underline{K}_{1-k/m}(M\ell) \approx \frac{\pi 2^{-k/m}}{\Gamma(k/m)}(M\ell)^{k/m-1} \quad . \tag{4.4}$$

If the n-th loop shrinks we have, therefore,

$$\mathcal{L}^{-1}\left[P_n(\theta_1, \cdots, \theta_n)\right] \to \quad \propto \mathcal{L}^{-1}\left[P_{n-1}(\theta_1, \cdots, \theta_{n-1})\right] \quad . \tag{4.5}$$

Then in the limit $M\ell_n \to 0$, $\tilde{\Delta}_n(\theta_1, \cdots, \theta_n)$ must satisfy the following relation:

$$\mathcal{L}^{-1}\left[\tilde{\Delta}_n(\theta_1, \cdots, \theta_n)\right] \to \quad \propto \mathcal{L}^{-1}\left[\frac{\partial}{\partial \Lambda}\tilde{\Delta}_{n-1}(\theta_1, \cdots, \theta_{n-1})\right] \tag{4.6}$$

This relation restricts the possible form of $\tilde{\Delta}_n(\theta_1, \cdots, \theta_n)$.

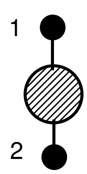
As we want $\tilde{\Delta}_n(\theta_1, \cdots, \theta_n)$ to be expressed as a polynomial of P_j and their derivatives with respect to Λ, we need to introduce a notation

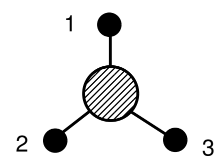
$$[\mathcal{S}P_{1,\cdots,i_1}P_{j_2,\cdots,j_2+i_2-1}\cdots P_{n-i_\ell+1,\cdots,n}]\,(\theta_1,\theta_2,\cdots\theta_n) \equiv \tag{4.7}$$
$$\frac{1}{n!}\sum_{\sigma\in\mathcal{P}_n} P_{i_1}(\theta_{\sigma(1)},\cdots,\theta_{\sigma(i_1)})P_{i_2}(\theta_{\sigma(j_2)},\cdots,\theta_{\sigma(j_2+i_2-1)})\cdots P_{i_\ell}(\theta_{\sigma(n-i_\ell+1)},\cdots,\theta_{\sigma(n)})\ ,$$

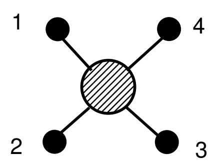
where $\mathcal{P}_n$ represents the permutations of $(1,2,\cdots,n)$. To be more specific, for example

$$[\mathcal{S}P_{123}P_{234}]\,(\theta_1,\theta_2,\theta_3,\theta_4) = \frac{1}{4!}\sum_{\sigma\in\mathcal{P}_4} P_3(\theta_{\sigma(1)},\theta_{\sigma(2)},\theta_{\sigma(3)})P_3(\theta_{\sigma(2)},\theta_{\sigma(3)},\theta_{\sigma(4)})$$
$$[\mathcal{S}P_{1234}P_{34}]\,(\theta_1,\theta_2,\theta_3,\theta_4) = \frac{1}{4!}\sum_{\sigma\in\mathcal{P}_4} P_4(\theta_{\sigma(1)},\theta_{\sigma(2)},\theta_{\sigma(3)}\theta_{\sigma(4)})P_2(\theta_{\sigma(3)},\theta_{\sigma(4)}) \quad . \tag{4.8}$$

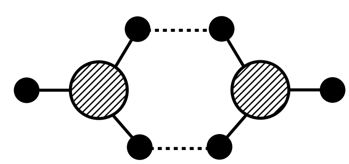
It is convenient to represent $P_n(\theta_1, \cdots, \theta_n)$ by an n-vertex which connects n external legs. For example for $n = 2$, 3 and 4

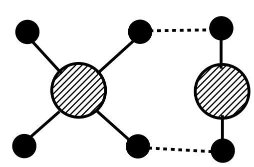
$P_2(\theta_1,\theta_2) \equiv$ 

, $P_3(\theta_1,\theta_2,\theta_3) \equiv$ 

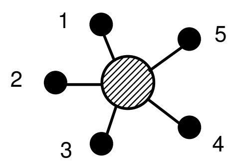
$P_4(\theta_1,\theta_2,\theta_3,\theta_4) \equiv$  (4.9)

The n-vertex can be regarded as a dual graph of the n-gon (polygon) which corresponds to $\mathcal{D}_n$. In terms of these vertices, let us express eq. (4.8) as follows.

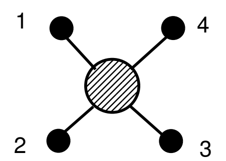
$[\mathcal{S}P_{123}P_{234}]\,(\theta_1,\theta_2,\theta_3,\theta_4) \ \equiv$ 

$[\mathcal{S}P_{1234}P_{34}]\,(\theta_1,\theta_2,\theta_3,\theta_4) \ \equiv$  (4.10)

The relation eq. (4.5) can be represented, for example, as



$$\to \quad \propto$$



(4.11)

in the case of n=5.

Now we are concerned with the case of n=4 first. Let us recall that for n=3

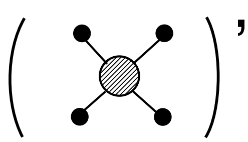
$$\tilde{\Delta}_3(\theta_1,\theta_2,\theta_3) \propto P_3(\theta_1,\theta_2,\theta_3) \quad . \tag{4.12}$$

$\tilde{\Delta}_4(\theta_1,\theta_2,\theta_3,\theta_4)$ must include a term which becomes proportional to $P_3(\theta_1,\theta_2,\theta_3)$ in the limit $M\ell_4 \to 0$, which is $P_4(\theta_1,\theta_2,\theta_3,\theta_4)$. $\tilde{\Delta}_4(\theta_1,\theta_2,\theta_3,\theta_4)$ may also include terms which vanish in this limit. Such terms must consist of the product of two multi-vertices which have 6 external legs in total.

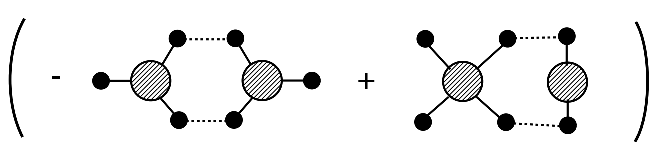
By explicit computation, we find

$$\frac{-1}{2!}\tilde{\Delta}_4(\theta_1,\theta_2,\theta_3,\theta_4) \mid = \frac{\partial}{\partial\Lambda}P_4(\theta_1,\theta_2,\theta_3,\theta_4) - \left[\mathcal{S}P_{123}P_{234}\right](\theta_1,\theta_2,\theta_3,\theta_4)$$
$$+\left[\mathcal{S}P_{1234}P_{34}\right](\theta_1,\theta_2,\theta_3,\theta_4)$$

$$=$$



$$+$$



$$, \tag{4.13}$$

where the prime represents the differentiation with respect to Λ.

Carrying out the procedure indicated in (1.3) for the case $n=4$ together with eqs. (2.28), (4.13), we obtain the complete answer for the macroscopic four loop ampitude: [6]

$$\mathcal{A}_4(\ell_1,\cdots\ell_4) = \mathcal{A}_4^{fusion}(\ell_1\cdots\ell_4)$$

[6] This was briefly reported in [17].

$$+ \quad \frac{1}{m^2(m+1)^2} t^{-2-\frac{1}{m}} \left(\prod_{j=1}^{4} \frac{M\ell_j}{\pi} \right) \left(\frac{1}{4!} \sum_{\sigma \in \mathcal{P}_4} \right) \Bigg\{ - \sum_{\mathcal{D}_3} \sum_{\mathcal{D}'_3} \prod_{j=1}^{4} \left[B_j^{123} * B_j'^{234} \right] \left(M\ell_{\sigma(j)} \right)$$

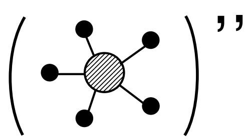
$$+ \quad \sum_{\mathcal{D}_4} \sum_{\mathcal{D}'_2} \prod_{j=1}^{4} \left[B_j^{1234} * B_j'^{34} \right] \left(M\ell_{\sigma(j)} \right) \Bigg\} \quad . \tag{4.14}$$

Here $B_j^{123} = (\underline{K}_{1-k_1/m}, \underline{K}_{1-k_2/m}, \underline{K}_{1-k_3/m}, 1)$, $B_j'^{234} = (1, \underline{K}_{1-k'_2/m}, \underline{K}_{1-k'_3/m}, \underline{K}_{1-k'_4/m})$, $B_j^{1234} = (\underline{K}_{1-k_1/m}, \underline{K}_{1-k_2/m}, \underline{K}_{1-k_3/m}, \underline{K}_{1-k_4/m})$ and $B_j'^{34} = (1, 1, \underline{K}_{1-k'_3/m}, \underline{K}_{1-k'_4/m})$. We have introduced $\mathcal{D}_3 \equiv \mathcal{D}_3(k_1-1, k_2-1, k_3-1)$, $\mathcal{D}'_3 \equiv \mathcal{D}_3(k'_1-1, k'_2-1, k'_3-1)$, $\mathcal{D}_4 \equiv \mathcal{D}_4(k_1-1, k_2-1, k_3-1, k_4-1)$ and $\mathcal{D}'_2 \equiv \mathcal{D}_2(k'_1-1, k'_2-1)$ and have defined the convolution $A * B(M\ell)$ by

$$[A * B]\,(M\ell) \equiv \int_0^{\ell} \frac{M d\ell'}{\pi} A(M\ell') B(M(\ell - \ell')) \quad . \tag{4.15}$$

Let us now turn to the n=5 case. $\tilde{\Delta}_5(\theta_1, \cdots, \theta_5)$ include a term which is proportional to

$$\frac{\partial^2}{\partial \Lambda^2} P_5(\theta_1, \cdots, \theta_5) =$$

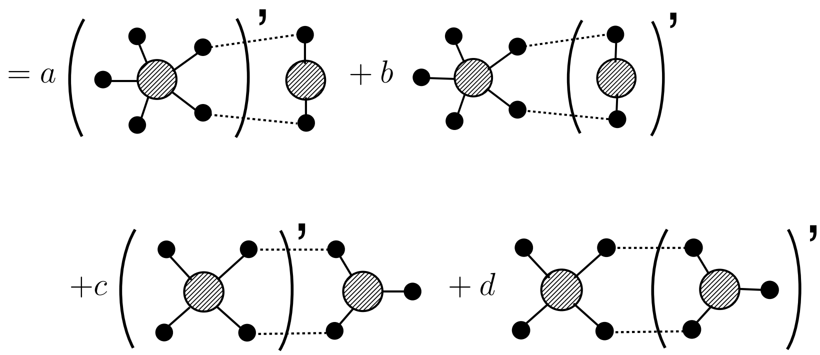


(4.16)

corresponding to the first term in eq. (4.13). Corresponding to the second term in eq. (4.13), $\tilde{\Delta}_5(\theta_1, \cdots, \theta_5)$ must include a term which consists of the product of two multi-vertices with 7 extenal legs in total. The possible form is

$$a\Big[\mathcal{S}\frac{\partial}{\partial\Lambda}(P_{12345})P_{45}\Big] + b\Big[\mathcal{S}P_{12345}\frac{\partial}{\partial\Lambda}(P_{45})\Big]$$

$$+c\Big[\mathcal{S}\frac{\partial}{\partial\Lambda}(P_{1234})P_{345}\Big] + d\Big[\mathcal{S}P_{1234}\frac{\partial}{\partial\Lambda}(P_{345})\Big]$$



. (4.17)

In the limit $M\ell_5 \to 0$, it becomes

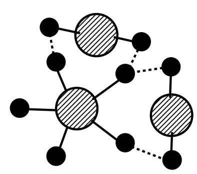
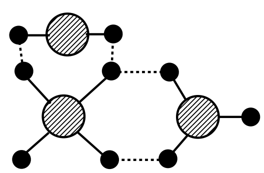
$$(3a+c)\Big[\mathcal{S}\frac{\partial}{\partial\Lambda}(P_{1234})P_{34}\Big] + (3b+d)\Big[\mathcal{S}P_{1234}\frac{\partial}{\partial\Lambda}(P_{34})\Big]$$
$$+2c\Big[\mathcal{S}\frac{\partial}{\partial\Lambda}(P_{123})P_{234}\Big] + 2d\Big[\mathcal{S}P_{123}\frac{\partial}{\partial\Lambda}(P_{234})\Big] \quad . \tag{4.18}$$

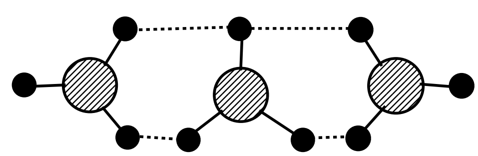
We require this expression to be proportional to the Λ-derivative of the second term in eq .(4.13). We find

$$a = (1-c)/3, \quad b = (2+c)/3, \quad d = -1-c \quad . \tag{4.19}$$

$\tilde{\Delta}_5(\theta_1, \cdots, \theta_5)$ may include terms which vanish in the limit under consideration as well. They must consist of the products of three multi-vertices with 9 external legs in total. As one of the such terms we have

$$[\mathcal{S}P_{1234}P_{34}P_{45}] - [\mathcal{S}P_{1234}P_{23}P_{345}] + [\mathcal{S}P_{123}P_{124}P_{235}]$$

$=$  $-\,2$ 

$+$  . (4.20)

(There are some other combinations which satisfy the conditions.) $\tilde{\Delta}_5(\theta_1, \cdots, \theta_5)$ must be expressed as a linear combination of the above three types of terms eq. (4.16), eq. (4.17) and eq. (4.20) if the assumption under consideration is true. By explicit calculation, we have found in fact that $\tilde{\Delta}_5(\theta_1, \cdots, \theta_5)$ can be expressed as a linear combination of eq. (4.16), eq. (4.17) and eq. (4.20) :

$$\frac{-1}{3!}\tilde{\Delta}_5(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5) = \left(\frac{\partial}{\partial\Lambda}\right)^2 P_5(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5)$$
$$+ \quad \left[\mathcal{S}P_{12345}\left(2\frac{\overrightarrow{\partial}}{\partial\Lambda} + 3\frac{\overleftarrow{\partial}}{\partial\Lambda}\right)P_{45}\right](\theta_1,\theta_2,\theta_3,\theta_4,\theta_5)$$
$$- \quad \left[\mathcal{S}P_{1234}\left(\frac{\overrightarrow{\partial}}{\partial\Lambda} + 4\frac{\overleftarrow{\partial}}{\partial\Lambda}\right)P_{345}\right](\theta_1,\theta_2,\theta_3,\theta_4,\theta_5)$$
$$+ \quad \left[(2P_{1234}P_{34}P_{45} - 4P_{123}P_{23}P_{345} + 2P_{123}P_{124}P_{235})\right](\theta_1,\theta_2,\theta_3,\theta_4,\theta_5) \; . \tag{4.21}$$

Following the same procedure as obtaining eq. (4.14), we find the complete answer for the five loop amplitude:

$$
\begin{aligned}
&\mathcal{A}_5(\ell_1,\cdots\ell_5) = \mathcal{A}_5^{fusion}(\ell_1\cdots\ell_5) + \frac{1}{m^2(m+1)^3}\left(\prod_{j=1}^{5}\frac{M\ell_j}{\pi}\right)\left(\frac{1}{5!}\sum_{\sigma\in\mathcal{P}_5}\right) \\
\times\ & \Bigg[\left(2t^{-1-\frac{3}{2m}}\left(\frac{\partial}{\partial t}t^{-1}\right)_R + 3t^{-1}\left(\frac{\partial}{\partial t}t^{-1-\frac{3}{2m}}\right)_L\right)\sum_{\mathcal{D}_5}\sum_{\mathcal{D}'_2}\prod_{j=1}^{5}\left[B_j^{12345} * B_j'^{45}\right](M\ell_{\sigma(j)}) \\
-\ & \left(t^{-1-\frac{1}{m}}\left(\frac{\partial}{\partial t}t^{-1-\frac{1}{2m}}\right)_R + 4t^{-1-\frac{1}{2m}}\left(\frac{\partial}{\partial t}t^{-1-\frac{1}{m}}\right)_L\right)\sum_{\mathcal{D}_4}\sum_{\mathcal{D}'_3}\prod_{j=1}^{5}\left[B_j^{1234} * B_j'^{345}\right](M\ell_{\sigma(j)}) \\
-\ & \frac{1}{m}t^{-3-\frac{3}{2m}}\Bigg(2\sum_{\mathcal{D}_5}\sum_{\mathcal{D}'_2}\sum_{\mathcal{D}''_2}\prod_{j=1}^{5}\left[B_j^{12345} * B_j'^{34} * B_j''^{45}\right](M\ell_{\sigma(j)}) \\
-\ & 4\sum_{\mathcal{D}_4}\sum_{\mathcal{D}'_2}\sum_{\mathcal{D}''_3}\prod_{j=1}^{5}\left[B_j^{1234} * B_j'^{23} * B_j''^{345}\right](M\ell_{\sigma(j)}) \\
+\ & 2\sum_{\mathcal{D}_3}\sum_{\mathcal{D}'_3}\sum_{\mathcal{D}''_3}\prod_{j=1}^{5}\left[B_j^{123} * B_j'^{124} * B_j''^{235}\right](M\ell_{\sigma(j)})\Bigg)\Bigg] \quad . \qquad (4.22)
\end{aligned}
$$

Here $\left(\frac{\partial}{\partial t}\right)_{L,(R)}$ means the derivative acting only on the left(right) part of the convolutions. The rest of the notations here are similar to those of the $n=4$ case and will be self-explanatory. [7]

We conjecture that $\tilde{\Delta}_n(\theta_1,\cdots,\theta_n)$ can be represented as a polynomial of $P_j(\theta_1,\cdots,\theta_j)$ and their t derivatives for any n : the final answer would then be obtained by convolutions of various B_j's and their derivatives. If the conjecture is true in fact, the power counting argument tells us that the j-th term of $\tilde{\Delta}_n$ turns out to be represented by a figure which consists of the products of j multi-vertices with $n+2j$ external legs in total. We hope that, for higher loops, $\tilde{\Delta}_n(\theta_1,\cdots,\theta_n)$ can be put in principle in a form such as eqs. (4.14), (4.22) in the same manner as we have determined the four and five loops from the lower ones. Explicit determination of the full amplitude in this way beyond five loops, however, appears to us still very formidable.

[7] It is just a matter of writing to take a small length limit of eqs. (4.14),(4.22) to obtain the corresponding expression for the microscopic operators. (Use eq. (3.24)).

V. Acknowledgements

We thank Atsushi Ishikawa for enjoyable collaboration in [8, 9]. We also thank Kenji Hamada, Keiji Kikkawa, and Alyosha Morozov for helpful discussions on this subject.

Appendix A

In this appendix, we prove the recursion relations for

$$\begin{pmatrix} m_1, & m_2, & m_3, & \cdots \\ i_1, & i_2, & i_3, & \cdots \end{pmatrix}_n \qquad \text{with} \quad \sum_\ell m_\ell \leq n-2 \tag{A.1}$$

introduced in the text. The proof goes by mathematical inductions.

We will first prove the simplest case

$$\begin{pmatrix} m, & 0, & \cdots, & 0 \\ i_1, & i_2 & \cdots, & i_n \end{pmatrix}_n = \frac{1}{\prod\limits_{j(\neq i_1)}^{n} [i_1 - j]^2} \quad , \qquad \text{for} \quad m = n-2 \tag{A.2}$$

and

$$\begin{pmatrix} m, & 0, & \cdots, & 0 \\ i_1, & i_2 & \cdots, & i_n \end{pmatrix}_n = 0 \quad , \qquad \text{for} \quad m \leq n-3 \quad . \tag{A.3}$$

Assume that eq. (A.2) and eq. (A.3) are true at n. Without loss of generality, i_1 can be taken to be 1. Let us consider the left hand side of eq. (A.2) or eq. (A.3) in which n is replaced by $n+1$. To compute them we observe that the elements of $\mathcal{S}_{n+1}$ are generated by associating n different ways of inserting $[n+1]$ with each element $\sigma \in \mathcal{S}_n$. In the case where $[n+1]$ is inserted in between [1] and $[\sigma(1)]$, this contribution is equal to

$$-\sum_{\sigma\in\mathcal{S}_n} \frac{1}{[1-\sigma(1)]^m} \, \frac{[1-\sigma(1)]^m}{[1-(n+1)]^{m+1}[(n+1)-\sigma(1)]} \prod_{j=2}^{n} \frac{1}{[j-\sigma(j)]} \quad . \tag{A.4}$$

The contributions from the sum of the remaining $n-1$ insertions are found to be equal to

$$-\sum_{\sigma\in\mathcal{S}_n} \frac{1}{[1-\sigma(1)]^m} \, \frac{1}{[\sigma(1)-(n+1)][1-(n+1)]} \prod_{j=2}^{n} \frac{1}{[j-\sigma(j)]} \quad . \tag{A.5}$$

Here we have used

$$\frac{1}{[j-(n+1)][(n+1)-m]} = \frac{1}{[j-m]} \left(\frac{1}{[j-(n+1)]} - \frac{1}{[m-(n+1)]} \right) \quad . \tag{A.6}$$

Note also that $\prod_{j=1}^{n-1} 1/[\sigma^j(1) - \sigma^{j+1}(1)] = \prod_{j=2}^{n} 1/[j - \sigma(j)]$. Putting eqs. (A.4) and (A.5) together, we find

$$\begin{pmatrix} m, & 0, & \cdots, & 0 \\ 1, & 2, & \cdots, & n+1 \end{pmatrix}_{n+1} =$$

$$- \sum_{\sigma \in \mathcal{S}_n} \frac{1}{[1-\sigma(1)]^m} \frac{\left\{1 - \left(\frac{[1-\sigma(1)]}{[1-(n+1)]}\right)^m\right\}}{[\sigma(1) - (n+1)][1-(n+1)]} \prod_{j=2}^{n} \frac{1}{[j-\sigma(j)]} \quad . \tag{A.7}$$

Factorizing the expression inside the bracket $\{ \ \cdots \ \}$, we have

$$\begin{pmatrix} m, & 0, & \cdots, & 0 \\ 1, & 2, & \cdots, & n+1 \end{pmatrix}_{n+1} = \sum_{l=1}^{m-1} \begin{pmatrix} m-l, & 0, & \cdots, & 0 \\ 1, & 2, & \cdots, & n \end{pmatrix}_n \frac{1}{[1-(n+1)]^{1+l}} \quad . \tag{A.8}$$

Then from the assumption, eq. (A.2) and eq. (A.3) are also satisfied when n is replaced by $n+1$. On the other hand for $n = 3$ eq. (A.2) and eq. (A.3) are clearly true, so we have proven the relations.

Now we turn to the more general case the proof of which is a straightforward generalization of the one given above. To derive the relations

$$\begin{pmatrix} m_1, & \cdots, & m_k, & 0, & \cdots, & 0 \\ i_1, & \cdots & i_k, & i_{k+1}, & \cdots, & i_n \end{pmatrix}_n = 0 \quad , \qquad \text{for} \quad \sum_{\ell=1}^{k} m_\ell \le n-3 \quad , \tag{A.9}$$

and

$$\begin{pmatrix} m_1, & \cdots, & m_k, & 0, & \cdots, & 0 \\ i_1, & \cdots & i_k, & i_{k+1}, & \cdots, & i_n \end{pmatrix}_n$$

$$= \sum_{j=1}^{k} \begin{pmatrix} m_1, & \cdots, & m_j - 1, & \cdots, & m_k, & 0, & \cdots, & 0 \\ i_1, & \cdots, & i_j, & \cdots, & i_k, & i_{k+1}, & \cdots, & i_{n-1} \end{pmatrix}_{n-1} \frac{1}{[j-n]^2} \quad ,$$

$$\text{for} \quad \sum_{\ell=1}^{k} m_\ell = n-2 \quad . \tag{A.10}$$

Let us assume eq. (A.9) at n .

We take $i_\ell = \ell, \quad \ell = 1 \sim k$ without loss of generality. The way in which the elements of $\mathcal{S}_{n+1}$ are generated is the same as the one given above. In the case where $[n+1]$ is inserted in between $[\ell]$ and $[\sigma(\ell)]$ $\ell = 1 \sim k$, the contribution is

$$- \sum_{\sigma \in \mathcal{S}_n} \frac{1}{[1-\sigma(1)]^{m_1+1}} \cdots \frac{1}{[\ell - \sigma(\ell)]^{m_\ell}} \frac{[\ell - \sigma(\ell)]^{m_\ell}}{[\ell - (n+1)]^{m_\ell+1}[(n+1) - \sigma(\ell)]}$$

$$\times \frac{1}{[(\ell+1)-\sigma(\ell+1)]^{m_{\ell+1}+1}} \cdots \frac{1}{[k-\sigma(k)]^{m_k+1}} \prod_{j(\neq 1,2,\cdots k)}^{n-1} \frac{1}{[j-\sigma(j)]} \quad . \quad \text{(A.11)}$$

The contributions from the sum of the remaining insertions are

$$\begin{aligned} & - \sum_{\sigma \in \mathcal{S}_n} \sum_{\ell(\neq p_2,\cdots p_k)}^{n-1} \frac{1}{[1-\sigma(1)]^{m_1+1}} \cdots \frac{1}{[k-\sigma(k)]^{m_{k+1}}} \\ & \times \prod_{j(\neq p_2,\cdots p_k)}^{n-1} \frac{1}{[\sigma^j(1)-\sigma^{j+1}(1)]} \left(\frac{1}{[\sigma^\ell(1)-(n+1)]} - \frac{1}{[\sigma^{\ell+1}(1)-(n+1)]} \right) \text{(A.12)} \end{aligned}$$

Here p_ℓ $\ell = 1 \sim k$ are such that $\sigma^{p_\ell}(1) = \ell$. Using eq. (A.6) again, we find that this equals

$$\begin{aligned} & - \sum_{\sigma \in \mathcal{S}_n} \sum_{\ell=1}^{k} \frac{1}{[1-\sigma(1)]^{m_1+1}} \cdots \frac{1}{[\ell-\sigma(\ell)]^{m_\ell}} \frac{1}{[\sigma(\ell)-(n+1)][\ell-(n+1)]} \\ & \frac{1}{[(\ell+1)-\sigma(\ell+1)]^{m_{\ell+1}+1}} \cdots \frac{1}{[k-\sigma(k)]^{m_k+1}} \prod_{j(\neq 1,2,\cdots k)}^{n-1} \frac{1}{[j-\sigma[j]]} \quad . \quad \text{(A.13)} \end{aligned}$$

Putting eqs. (A.11) and (A.13) together, we find

$$\begin{aligned} & \begin{pmatrix} m_1, & \cdots, & m_k, & 0, & \cdots, & 0 \\ 1, & \cdots & k, & k+1, & \cdots, & n+1 \end{pmatrix}_{n+1} \\ = & -\sum_{\ell}^{k} \sum_{\sigma \in \mathcal{S}_n} \frac{1}{[1-\sigma(1)]^{m_1+1}} \cdots \frac{1}{[\ell-\sigma(\ell)]^{m_\ell}} \frac{\left\{1-\left(\frac{[\ell-\sigma(\ell)]}{[\ell-(n+1)]}\right)^{m_\ell}\right\}}{[\sigma(\ell)-n][\ell-n]} \\ \times & \frac{1}{[(\ell+1)-\sigma(\ell+1)]^{m_{\ell+1}+1}} \cdots \frac{1}{[k-\sigma(k)]^{m_k+1}} \prod_{j(\neq 1,2,\cdots k)}^{n-1} \frac{1}{[j-\sigma(j)]} \quad . \text{(A.14)} \end{aligned}$$

Factorizing the expression inside the bracket, we have

$$\begin{aligned} & \begin{pmatrix} m_1, & \cdots, & m_k, & 0, & \cdots, & 0 \\ 1, & \cdots & k, & k+1 & \cdots, & n+1 \end{pmatrix}_{n+1} \\ & = \sum_{j=1}^{k} \sum_{l=1}^{m_j} \begin{pmatrix} m_1, & \cdots, & m_j - l, & \cdots, & m_k, & 0, & \cdots, & 0 \\ 1, & \cdots & j, & \cdots, & k, & k+1 & \cdots, & n \end{pmatrix}_n \frac{1}{[j-n]^{1+l}} \quad . \end{aligned}$$

(A.15)

Then from the assumption eq. (A.9) at n, eq. (A.9) in which n is replaced by $n+1$ is also satisfied. On the other hand for $n = 3$ eq. (A.9) is clearly satisfied, so we have proven eq. (A.9) and eq. (A.10) .

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